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# REPRESENTATIONS OF SEMI LATTICE IN FACTOR CONGRUENCE ON PRE A*- ALGEBRA

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**Abstract:** In this paper we define $\otimes$-Semi lattice on Pre A*-algebra $A$ and prove that for each $a \in C(A)$ define $\beta_a = \{ (x,y) \ / \ a\vee x = a\vee y\}$ is a factor congruence on $A$ and $\beta_{a^{\otimes}}$ is direct complement of $\beta_a$ and also prove that $\beta$ is a factor congruence on $A$ iff $\beta = \beta_x$, for some $x \in C(A)$.

**Keywords:** Pre A*-algebra, $\otimes$-Semi lattice, central element, factor congruence.

**AMS subject classification (2000):** 06E05, 06E25, 06E99, 06B10.

**Introduction:** In 1994, P. Koteswara Rao[2] first introduced the concept A*-Algebra $(A,\wedge,\vee,*,(-)^{\sim},(-)_{\pi},0,1,2)$ not only studied the equivalence with Ada, C-algebra, Ada's connection with 3-Ring, the If-Then-Else structure over A*-algebra and Ideal of A*-algebra. . In 2000, J. Venkateswara Rao [5] introduced the concept of Pre A*-algebra $(A,\wedge,\vee,(-)^{\sim})$ as the variety generated by the 3-element algebra A = {0,1,2} which is an algebraic form of three valued conditional logic.In [6] Satyanarayana et al. generated Semilattice structure on Pre A*-Algebras . In [7] Satyanarayana.A, et.all derive necessary and sufficient conditions for pre A*-algebra A to become a Boolean algebra in terms of the partial ordering.

**1. Preliminaries:** In this section we concentrate on the algebraic structure of Pre A*-algebra and state some results which will be used in the later text.

**1.1. Definition:** An algebra $(A,\wedge,\vee,(-)^{\sim})$ where A is a non-empty set with $1,\wedge,\vee$ are binary operations and $(-)^{\sim}$ is a unary operation satisfying

(a) $x^{\sim\sim}=x \quad \forall x\in A$
(b) $x\wedge x=x, \quad \forall x\in A$
(c) $x\wedge y=y\wedge x, \quad \forall x,y\in A$
(d) $(x\wedge y)^{\sim}=x^{\sim}\vee y^{\sim} \quad \forall x,y\in A$
(e) $x\wedge(y\wedge z)=(x\wedge y)\wedge z, \quad \forall x,y,z\in A$
(f) $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z), \quad \forall x,y,z\in A$
(g) $x\wedge y=x\wedge(x^{\sim}\vee y), \quad \forall x,y\in A$ is called a Pre A*-algebra.

1.2. **Example:** 3 = {0, 1, 2} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

| $\wedge$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 2 |
| 1 | 0 | 1 | 2 |
| 2 | 2 | 2 | 2 |

| $\vee$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 1 | 2 |
| 2 | 2 | 2 | 2 |

| $x$ | $x^{\sim}$ |
|---|---|
| 0 | 1 |
| 1 | 0 |
| 2 | 2 |

1.3. **Note:** The elements 0, 1, 2 in the above example satisfy the following laws:

(a) $2^{\sim} = 2$ (b) $1 \wedge x = x$ for all $x \in 3$

(c) $0 \vee x = x$ for all $x \in 3$ (d) $2 \wedge x = 2 \vee x = 2$ for all $x \in 3$.

1.4. **Example:** 2 = {0, 1} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

| $\wedge$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

| $\vee$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

| x | $x^{\sim}$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

1.5. **Note:**

(i) $(2, \vee, \wedge, (-)^{\sim})$ is a Boolean algebra. So every Boolean algebra is a Pre A* algebra.

(ii) The identities 1.2(a) and 1.2(d) imply that the varieties of Pre A*-algebras satisfies all the dual statements of 1.2(b) to 1.2(g).

1.6. **Definition:** Let A be a Pre A*-algebra. An element $x \in A$ is called a central element of A if $x \vee x^{\sim} = 1$ and the set $\{x \in A / x \vee x^{\sim} = 1\}$ of all central elements of A is called the centre of A and it is denoted by B (A).

1.7. **Theorem:**[6] Let A be a Pre A*-algebra with 1, then B (A) is a Boolean algebra with the induced operations $\wedge, \vee, (-)^{\sim}$

1.8. **Lemma:** [6] Every Pre A*-algebra with 1 satisfies the following laws

(a) $x \vee 1 = x \vee x^{\sim}$ (b) $x \wedge 0 = x \wedge x^{\sim}$

1.9. **Lemma:** [6] Every Pre A*-algebra with 1 satisfies the following laws.

(a) $x \wedge (x^{\sim} \vee x) = x \vee (x^{\sim} \wedge x) = x$

(b) $(x \vee x^{\sim}) \wedge y = (x \wedge y) \vee (x^{\sim} \wedge y)$

(c) $(x \vee y) \wedge z = (x \wedge z) \vee (x^{\sim} \wedge y \wedge z)$

1.10. **Definition:** A relation $\theta$ on a Pre A*- algebra $(A, \wedge, \vee, (-)^{\sim})$ is called congruence relation if

(i) $\theta$ is an equivalence relation

(ii) $\theta$ is closed under $\wedge, \vee, (-)^{\sim}$

**2. $\otimes$-Semi Lattice on Pre A*- Algebra**

2.1. **Theorem:**[6] Let A b e a Pre A*-algebra $< A, \vee >$ is a semilattice.

**2.2. Definition:** Let $A$ be a Pre A*-algebra with 2. If $\otimes$ is an unary operation on $A$ such that for any $x$, $y$, $a \in A$ satisfying the following

(i) $a \vee a^{\otimes} = 2$

(ii) $2^{\otimes} \vee x = x$

(iii) $a \vee ((a \vee x)^{\otimes} \vee (a^{\otimes} \vee y)^{\otimes})^{\otimes} = a \vee x$, $a^{\otimes} \vee ((a \vee x)^{\otimes} \vee (a^{\otimes} \vee y)^{\otimes})^{\otimes} = a^{\otimes} \vee y$

(iv) $((a \vee x)^{\otimes} \vee (a^{\otimes} \vee x)^{\otimes})^{\otimes} = x$

then $A$ is called a $\otimes$-Semi lattice.

**2.3. Lemma:** $x^{\otimes\otimes} = x$, for all $x \in A$

**Proof:** By 2.2 Definition (iii) we have $a^{\otimes} \vee ((a \vee y)^{\otimes} \vee (a^{\otimes} \vee x)^{\otimes})^{\otimes} = a^{\otimes} \vee x$

Let $a = 2$ then $2^{\otimes} \vee ((2 \vee y)^{\otimes} \vee (2^{\otimes} \vee x)^{\otimes})^{\otimes} = 2^{\otimes} \vee x$

$\Rightarrow ((2 \vee y)^{\otimes} \vee (2^{\otimes} \vee x)^{\otimes})^{\otimes} = x$ (by 2.2 Definition (ii))

$\Rightarrow (2^{\otimes} \vee x^{\otimes})^{\otimes} = x$

$\Rightarrow x^{\otimes\otimes} = x$

**2.4. Definition:** Let $A$ be a $\otimes$-Semi lattice. An element $a \in A$ is called central element if $a$ satisfies the following

(1) $a \vee x = a \vee y \Rightarrow a \vee x^{\otimes} = a \vee y^{\otimes}$

(2) $a^{\otimes} \vee x = a^{\otimes} \vee y \Rightarrow a^{\otimes} \vee x^{\otimes} = a^{\otimes} \vee y^{\otimes}$, for all $x$, $y \in A$

The set of all central elements is denoted by $C(A)$, and also observe that $a \in C(A)$ then $a^{\otimes} \in C(A)$

**2.5. Lemma:** Let $A$ be a $\otimes$-Semi lattice and $a \in C(A)$ then define $\beta_a = \{(x, y) / a \vee x = a \vee y\}$ is a congruence on $A$

**Proof:** Since $a \vee x = a \vee x$ then $(x, x) \in \beta_a$, the relation is reflexive.

Let $(x, y) \in \beta_a$ then $a \vee x = a \vee y \Rightarrow a \vee y = a \vee x$

$\Rightarrow (y, x) \in \beta_a$, the relation is symmetric

Let $(x, y) \in \beta_a$ and $(y, z) \in \beta_a$ then $a \vee x = a \vee y$ and $a \vee y = a \vee z \Rightarrow a \vee x = a \vee z$

$\Rightarrow (x, z) \in \beta_a$, the relation is transitive.

Hence the relation $\beta_a$ is equivalence relation.

Let $x, y, z, t \in A$ such that $(x, y) \in \beta_a$ and $(z, t) \in \beta_a$ then $a \vee x = a \vee y$, $a \vee z = a \vee t$

Now $a \vee (x \vee z) = (a \vee x) \vee z$

$= (a \vee x) \vee (a \vee z) = (a \vee y) \vee (a \vee t) = a \vee (y \vee t)$

This shows that $(x \vee z, y \vee t) \in \beta_a$

Hence $\beta_a$ is closed under $\vee$.

Let $x, y \in A$ such that $(x, y) \in \beta_a$ then $a \vee x = a \vee y \Rightarrow a\% \wedge x\% = a\% \wedge y\%$

$\Rightarrow a \vee x^{\otimes} = a \vee y^{\otimes}$ (since $a \in C(A)$)

$\Rightarrow (x^{\otimes}, y^{\otimes}) \in \beta_a$

Therefore $\beta_a$ is closed under $\otimes$

Therefore $\beta_a$ is a congruence relation on $A$.

**2.6. Lemma:**[6] Let A be a Pre A*-algebra define a relation $\leq$ on A by $x \leq y$ iff $x \vee y = y$ then (A, $\leq$) is a poset.

**2.7. Definition:** Let $A$ be a Pre A*-algebra and $\alpha \in \mathrm{Con(A)}$.Then $\alpha$ is called factor congruence if there exist $\beta \in \mathrm{Con(A)}$ such that $\alpha \,\mathrm{I}\, \beta = \Delta_A$ and $\alpha \, o \, \beta =$ A×A. In this case $\beta$ is called direct complement of $\alpha$.

**2.8. Lemma:** Let $A$ be a $\otimes$-Semi lattice and $a \in C(A)$ then $\beta_a$ is a factor congruence on $A$ and $\beta_{a^\otimes}$ is direct complement of $\beta_a$.

**Proof:** Let $(x, y) \in \beta_a \,\mathrm{I}\, \beta_{a^\otimes}$

$\Rightarrow (x, y) \in \beta_a$ and $(x, y) \in \beta_{a^\otimes}$

$\Rightarrow a \vee x = a \vee y$ and $a^\otimes \vee x = a^\otimes \vee y$

Now $y = ((a \vee y)^\otimes \vee (a^\otimes \vee y)^\otimes)^\otimes$ (by 2.2 Definition (iv))

$= ((a \vee y)^\otimes \vee (a^\otimes \vee y)^\otimes)^\otimes \vee y = ((a \vee x)^\otimes \vee (a^\otimes \vee x)^\otimes)^\otimes \vee y = x \vee y$

Therefore $x \vee y = y$.

Hence $x \leq y$

Similarly we can prove that $y \leq x$ and hence $x = y$.

Therefore $\beta_a \,\mathrm{I}\, \beta_{a^\otimes} = \Delta_A$

Let $x \neq y$ and $z = ((a \vee x)^\otimes \vee (a^\otimes \vee y)^\otimes)^\otimes$

Now $a \vee z = a \vee ((a \vee x)^\otimes \vee (a^\otimes \vee y)^\otimes)^\otimes = a \vee x$ (by 2.2 Definition (iii))

Therefore $(x, z) \in \beta_a$

Now $a^\otimes \vee z = a^\otimes \vee ((a \vee x)^\otimes \vee (a^\otimes \vee y)^\otimes)^\otimes = a^\otimes \vee y$ (by 2.2 Definition (iii))

Therefore $(z, y) \in \beta_{a^\otimes}$

Thus $(x, y) \in \beta_{a^\otimes} \, o \, \beta_a$

Therefore $A \times A \subseteq \beta_{a^\otimes} \, o \, \beta_a$

Clearly $\beta_{a^\otimes} \, o \, \beta_a \subseteq A \times A$

Hence $\beta_{a^\otimes} \, o \, \beta_a = A \times A$

Therefore $\beta_a$ is a factor congruence on $A$ and $\beta_{a^\otimes}$ is direct complement of $\beta_a$.

**2.9. Lemma:** Let $A$ be a $\otimes$-Semi lattice and $\beta$ is a congruence on $A$. Then $\beta$ is a factor congruence on $A$ iff $\beta = \beta_x$, for some $x \in C(A)$.

**Proof:** Suppose $\beta$ is a factor congruence on $A$ then there exists a $\theta$ such that $\beta \,\mathrm{I}\, \theta = \Delta_A$ and $\beta \, o \, \theta =$ A×A.

Now $(2, 2^\otimes) \in \beta \, o \, \theta$ then there exists $x \in A$ such that $(2, x) \in \theta$, $(x, 2^\otimes) \in \beta$

Now we show that $\beta = \beta_x$

Let (p, q) $\in \beta_x$ then x$\vee$ p= x$\vee$ q .

Since $(x, 2^\otimes) \in \beta$ we have (x$\vee$p, $2^\otimes \vee$ p), (x$\vee$q, $2^\otimes \vee$ q) $\in \beta$ that is (x$\vee$p, p), (x$\vee$q, q) $\in \beta$ which imply that (p,q) $\in \beta$.

Hence $\beta_x \subseteq \beta$.

Suppose (p,q) $\in \beta$ then (x$\vee$p, x$\vee$ q) $\in \beta$.

Since $(2,x)\in\theta$ we have $(2\vee p, x\vee p), (2\vee q, x\vee q)\in\theta$ that is $(2, x\vee p), (2, x\vee q)\in\theta$ which implies that $(x\vee p, x\vee q)\in\theta$

Therefore $(x\vee p, x\vee q)\in\beta \cap \theta=\Delta_A$ and hence $x\vee p=x\vee q \Rightarrow (p,q)\in\beta_x$

Hence $\beta\subseteq\beta_x$

Thus $\beta=\beta_x$.

Now we prove that $\theta=\beta_{x^\otimes}$

Let $(p,q)\in\beta_{x^\otimes}$ then $x^\otimes\vee p=x^\otimes\vee q$

Since $(2,x)\in\theta$ we have $(2^\otimes,x^\otimes)\in\theta \Rightarrow (2^\otimes\vee p, x^\otimes\vee p), (2^\otimes\vee q, x^\otimes\vee q)\in\theta$

$\Rightarrow (p, x^\otimes\vee p), (q, x^\otimes\vee q)\in\theta \Rightarrow (p,q)\in\theta$

Therefore $\beta_{x^\otimes}\subseteq\theta$

Let $(p,q)\in\theta$

Since $(x,2^\otimes)\in\beta$ we have $(x^\otimes,2)\in\beta$

$\Rightarrow (x^\otimes\vee p, 2\vee p), (x^\otimes\vee q, 2\vee q)\in\beta \Rightarrow (x^\otimes\vee p, 2), (x^\otimes\vee q, 2)\in\beta$

$\Rightarrow (x^\otimes\vee p, x^\otimes\vee q)\in\beta$

Since $(p,q)\in\theta$ we have $(x^\otimes\vee p, x^\otimes\vee q)\in\theta$

Therefore $(x^\otimes\vee p, x^\otimes\vee q)\in\beta\cap\theta=\Delta_A$

$\Rightarrow x^\otimes\vee p=x^\otimes\vee q \Rightarrow (p,q)\in\beta_{x^\otimes} \Rightarrow \theta\subseteq\beta_{x^\otimes}$

Therefore $\theta=\beta_{x^\otimes}$

Now we show that $x\in C(A)$.

Let $x\vee t=x\vee w$ that is $(t,w)\in\beta_x=\beta \Rightarrow (t^\otimes,w^\otimes)\in\beta$ (since $\beta$ is a congruence)

$\Rightarrow (t^\otimes,w^\otimes)\in\beta_x \Rightarrow x\vee t^\otimes=x\vee w^\otimes$

Let $x^\otimes\vee t=x^\otimes\vee w$

$\Rightarrow (t,w)\in\beta_{x^\otimes}=\theta \Rightarrow (t^\otimes,w^\otimes)\in\theta$ (since $\theta$ is a congruence)

$\Rightarrow (t^\otimes,w^\otimes)\in\beta_{x^\otimes} \Rightarrow x^\otimes\vee t^\otimes=x^\otimes\vee w^\otimes$

Therefore $x\in C(A)$.

Conversely suppose that $\beta=\beta_x$, for some $x\in C(A)$.

By 2.8. Lemma $\beta_x$ is a factor congruence.

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